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A similarity analysis of heat conduction in a wedge with or without phase change

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NOMENCLATURE

- G^c geometric factor of the wedge under pure heat conduction
- G^m geometric factor of the wedge with moving heat source
- K thermal conductivity (subscripts s for solid, L for liquid)
- q heat flow rate per unit area and time
- Q_Λ integrated heat flow rate per unit time over Λ
- Q^c_Λ integrated heat flow rate per unit time over Λ in the pure heat conduction problem
- Q^m_Λ integrated heat flow rate per unit time over Λ in the moving source problem
- r radial coordinate
- r^* dimensionless radial coordinate, $r/(4\alpha t)^{1/2}$
- R interface position between liquid and solid
- t time
- T_i initial temperature
- T_w wall temperature
- T_s temperature of solid
- T_L temperature of liquid
- T_F fusion temperature
- T^c normalized temperature in the pure heat conduction problem
- T^m normalized temperature in the moving source problem.

Greek symbols

- α thermal diffusivity
- β ratio of latent heat to sensible heat, $L/c_p(T_F - T_w)$
- Δ wedge invariant in the pure heat conduction problem
- Δ^p wedge invariant in the phase change heat conduction problem
- η dimensionless interface position between liquid and solid, $R/\sqrt{(4\alpha t)}$
- θ angular coordinate
- θ_0 wedge angle
- Λ an arbitrary r^* at the wedge surface
- λ interface constant in one-dimensional Stefan problem.

INTRODUCTION

THE HEAT conduction in an infinite wedge enclosure was solved analytically in 1942 by Jaeger [1]. For the case of uniform initial temperature and constant surface temperature on the wedge surface Jaeger gave a solution of the temperature distribution involving two parameters, the angular coordinate θ and the dimensionless radial coordinate $r^* = r/(4\alpha t)^{1/2}$. In 1973 Budhia and Kreith [2] considered the same problem with melting or freezing, as an extension of Rathjen and Jiji's work [3] on a rectangular wedge enclosure. The solution of this nonlinear problem has been given in an infinite integral series of the same independent variables θ and r^*. In 1982 Wei and Berry [4], investigating the heat conduction problem in a wedge without change of phase, showed that the difference between the rate of heat loss per unit time and depth along the wedge surface and that from a corresponding surface of a corner-free, semi-infinite solid is independent of time. The term 'wedge invariant' was introduced to define this invariability.

This paper shows that, for the above-mentioned transient heat conduction problem with or without phase change, the integrated heat flow rate Q_Λ through an arbitrarily chosen dimensionless wedge surface Λ remains constant at all times, and therefore is independent of time.

Based on the invariability of the heat flow rate itself the 'wedge invariant', introduced by Wei and Berry for the case of pure heat conduction, can now be extended to heat conduction with change of phase. Simplified formulas will also be suggested for calculating the integrated heat flow rate over the wedge surface.

ANALYSIS

Heat conduction in a wedge without change of phase

In cylindrical coordinates, the constant property heat conduction problem in a wedge is governed by

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1}$$

with

$$T(r, 0, t) = 1$$
$$T(r, \theta_0, t) = 1$$
$$T(r, \theta, 0) = 0.$$

The analytical solution of equation (1) is given by Jaeger [1] as

$$T(r,\theta,t) = 1 - \frac{r}{\theta_0}\sum_{n=0}^{\infty} \sin\frac{(2n+1)\pi\theta}{\theta_0} \times \int_0^\infty e^{-(u^2(\alpha t/r^2))}\frac{J_v(u)\,du}{u} \tag{2}$$

where J_v is the Bessel function of the first kind of order v and $v = (2n+1)\pi/\theta_0$. By introducing a dimensionless similarity variable $r^* = r/(4\alpha t)^{1/2}$ the above solution can be written as a function of two variables, r^* and θ

$$T(r^*,\theta) = 1 - \frac{4}{\theta_0}\sum_{n=0}^{\infty} \sin\frac{(2n+1)\pi\theta}{\theta_0}\int_0^\infty e^{-(u/2r^*)^2}\frac{J_v(u)}{u}\,du. \tag{3}$$

The local heat flux along the wedge surface can be obtained as

$$q = -\frac{K}{r}\frac{\partial T}{\partial \theta}\bigg|_{\theta=0} = \frac{4K}{r\theta_0} \times \sum_{n=0}^{\infty}\frac{(2n+1)\pi}{\theta_0}\int_0^\infty e^{-(u/2r^*)^2}\frac{J_v(u)\,du}{u}. \tag{4}$$

The above expression does not provide similarity in r and t, however, integrating q along a unit-strip area, from $r = 0$ to an arbitrary $r = r'$, recovers the similarity. This integrated q is denoted by Q^c_Λ, where $\Lambda = r'/(4\alpha t)^{1/2}$, and

$$Q^c_\Lambda = \frac{4K}{\theta_0}\sum_{n=0}^{\infty}\frac{(2n+1)\pi}{\theta_0}\int_0^\Lambda\int_0^\infty e^{-(u/2r^*)^2}\frac{J_\nu(u)\,du}{u}\frac{dr^*}{r^*}. \quad (5)$$

As a consequence of equation (5), the heat flow rates at different times, $t_1, t_2, \ldots, t_i$ through corresponding wedge surfaces, $r_1, r_2, \ldots, r_i$ are identical, i.e. for $\Lambda_i = r_i/(4\alpha t_i)^{1/2} = \Lambda$

$$Q^c_{\Lambda_i} = Q^c_\Lambda. \quad (6)$$

Because of the previous result, the formula for calculating the integrated heat flow rate through an arbitrarily chosen dimensionless wedge surface parameter Λ can be written as

$$Q^c_\Lambda = KG^c(\Lambda, \theta_0) \quad (7)$$

where

$$G^c(\Lambda, \theta_0) = \frac{4\pi}{\theta_0^2}\sum_{n=0}^{\infty}(2n+1)\int_0^\Lambda\int_0^\infty e^{-(u/2r^*)^2}\frac{J_\nu(u)\,du}{u}\frac{dr^*}{r^*}. \quad (8)$$

The coefficient $G^c(\Lambda, \theta_0)$ can be regarded as a geometric factor. It depends on the dimensionless wedge surface Λ and the wedge angle θ_0 and can be evaluated.

The existence of the wedge invariant [4] can be explained in the light of the above analysis. The definition of the wedge invariant can be stated mathematically as

$$\Delta = \frac{1}{K}\left(\int_0^\infty q_{2d}\,dr - \int_0^\infty q_{1d}\,dr\right) \quad (9)$$

where the two-dimensional local heat flux q_{2d} is equivalent to equation (4), and the one-dimensional heat flux is the well-known expression [5]

$$q_{1d} = \frac{K}{\sqrt{(\pi\alpha t)}}. \quad (10)$$

For an arbitrarily chosen large enough dimensionless wedge surface parameter Λ_1, the heat conduction at $r^* > \Lambda_1$ can be considered one-dimensional. Therefore, equation (9) can be simplified to

$$\Delta = G^c(\Lambda_1, \theta_0) - \int_0^{\Lambda_1\sqrt{(4\alpha t)}}\frac{1}{\sqrt{(\pi\alpha t)}}dr$$

or

$$\Delta = \frac{Q^c_{\Lambda_1}}{K} - \frac{2}{\sqrt{\pi}}\Lambda_1. \quad (11)$$

Note that the above derivation is equivalent to carrying out the integrations to infinity. Obviously, in equation (11) both terms $Q^c_{\Lambda_1}/K$ and $(2/\sqrt{\pi})\Lambda_1$ are independent of time, thus there exists a wedge invariant.

Heat conduction in a wedge with phase change

The differential equations in the case of equal thermal diffusivities in liquid and solid phases can be stated as follows

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \quad (12)$$

$$T(r, 0, t) = T_w$$
$$T(r, \theta_0, t) = T_w$$
$$T(r, \theta, 0) = T_i$$
$$T(\infty, \theta, t) = T_i.$$

At the interface between liquid and solid

$$T = T_s = T_L = T_F, \quad 0 < \theta < \theta_0, \quad r = R(\theta, t), \quad t > 0,$$

and

$$\left(K_s\frac{\partial T_s}{\partial r} - K_L\frac{\partial T_L}{\partial r}\right)\left[1 + \frac{1}{R^2}\left(\frac{\partial R}{\partial \theta}\right)^2\right] = \rho L\frac{\partial R}{\partial t}.$$

Budhia and Kreith [2] tackled the above problem, as a special case, by normalizing the temperature variables in solid and liquid phases with respect to the fusion temperature as

$$T^* = T^*_s = \frac{T_s - T_F}{T_F - T_w} \quad \text{and} \quad T^* = T^*_L = \frac{K_L(T_L - T_F)}{K_s(T_F - T_w)}$$

and decomposing the phase change problem into a pure heat conduction problem and a moving heat source problem, i.e.

$$T^*(r, \theta, t) = T^c(r, \theta, t) + T^m(r, \theta, t) \quad (13)$$

with the conditions

$$T^c(r, 0, t) = -1$$
$$T^c(r, \theta_0, t) = -1 \quad (14)$$
$$T^c(r, \theta, 0) = T^*_i$$

and

$$T^m(r, 0, t) = 0$$
$$T^m(r, \theta_0, t) = 0 \quad (15)$$
$$T^m(r, \theta, 0) = 0.$$

It follows that the similarity of the integrated heat flow rate across the boundary surface, as demonstrated in the previous section for the pure heat conduction problem, exists in the phase change problem if the same similarity can be derived for the moving heat source problem.

The temperature distribution of the moving source problem given by Budhia and Kreith [2] is

$$T^m(r, \theta, t) = T^m(r^*, \theta) = \frac{4\beta}{\theta_0}\sum_{p'}\sin p'\theta \times \int_0^1\frac{d\tau}{1-\tau}\int_0^{\theta_0/2}\eta^2\exp\left(\frac{-\eta^2\tau + r^{*2}}{1-\tau}\right) \times I_{p'}\left(\frac{2\eta(\sqrt{\tau})r^*}{1-\tau}\right)\sin p'\theta'\,d\theta' \quad (16)$$

where $p' = n\pi/\theta_0$, $n = 1, 2, 3 \ldots \infty$, $I_{p'}$ denotes the modified Bessel function of the first kind, and η, the dimensionless interface position, is a function of the angular variable θ' and parameters β, λ and T^*_i. Accordingly the local heat flux can be obtained as

$$q^m = -(T_F - T_w)\frac{K_s}{r}\frac{\partial T^m}{\partial \theta}\bigg|_{\theta=0} = -(T_F - T_w)\frac{4K_s\beta}{r\theta_0} \times \sum_{p'}p'\int_0^1\frac{d\tau}{1-\tau}\int_0^{\theta_0/2}\eta^2\exp\left(\frac{-\eta^2\tau + r^{*2}}{1-\tau}\right) \times I_{p'}\left(\frac{2\eta(\sqrt{\tau})r^*}{1-\tau}\right)\sin p'\theta'\,d\theta'. \quad (17)$$

The integrated heat flow rate through the dimensionless wedge surface Λ can be similarly expressed as

$$Q^m_\Lambda = -K_s(T_F - T_w)G^m \quad (18)$$

where

$$G^m = \frac{4\beta}{\theta_0^2}\sum_{n=0}^{\infty}n\pi\int_0^\Lambda\int_0^1\frac{d\tau}{1-\tau}\int_0^{\theta_0/2} \times \eta^2\exp\left(\frac{-\eta^2\tau + r^{*2}}{1-\tau}\right) \times I_{p'}\left(\frac{2\eta(\sqrt{\tau})r^*}{1-\tau}\right) \times \sin\left(\frac{n\pi}{\theta_0}\theta'\right)d\theta'\frac{dr^*}{r^*} \quad (19)$$

and G^m is a geometric factor of the wedge in the moving source problem. It depends on the wedge parameters θ_0 and Λ, also on the parameters β, λ and T^*_i. G^m can be computed numerically and tabulated.

Expressions (18) and (19) state that, for the moving source problem, the integrated heat flow rate Q^m_Λ over a dimensionless wedge surface Λ, per unit thickness, is constant for a given system. The total integrated heat flow rate into the wedge enclosure for the phase change problem can be written as

$$Q_\Lambda = -K_s G^c(T_F - T_w)(1 + T_i^*) - K_s G^m(T_F - T_w). \quad (20)$$

The wedge invariability for the phase change heat conduction problem may be shown in a manner parallel to that for pure conduction

$$\Delta^p = \frac{1}{K}\left(\int_0^\infty q_{2d}\,dr - \int_0^\infty q_{1d}\,dr\right) \quad (21)$$

where q_{1d} is the heat flux in the one-dimensional Stefan problem [5]

$$q_{1d} = -\frac{K(T_F - T_w)}{(\operatorname{erf}\lambda)\sqrt{(\pi\alpha t)}}. \quad (22)$$

Correspondingly, equation (21) can be simplified to

$$\Delta^p = \frac{Q_{\Lambda_1}}{K} + \frac{2\Lambda_1(T_F - T_w)}{(\operatorname{erf}\lambda)\sqrt{\pi}}. \quad (23)$$

In equations (21)–(23) $K = K_s$ for freezing and $K = K_L$ for melting. It follows that both terms in equation (23) are independent of time, thus there exists a wedge invariant for the phase change heat conduction problem. It ought to be noted that the above derivation is contingent upon the approximate solution (16) given by Budhia and Kreith [2].

EVALUATION AND VALIDATION OF THE GEOMETRIC FACTORS

As a particular example, the geometric factors G^c and G^m for a rectangular corner, when $\beta = 1$ and $T_i^* = 0$, have been calculated. A Gaussian quadrature formula [6] was used to yield approximate solutions and the results are plotted in Fig. 1. It is notable that geometric factors G^c and G^m exhibit linear dependence on the dimensionless wedge surface parameter Λ except at the region near point $\Lambda = 0$, where the two-dimensional corner effect prevails.

It follows from equation (20) that, for $T_i^* = 0$, the sum of G^c and G^m can be regarded as the geometric factor for the total integrated heat flow rate into the wedge enclosure. In order to validate the presented similarity, a finite-difference solution for the temperature field and integrated heat flow rates of this particular problem was performed [7]. In Fig. 1 the numerical solutions are represented by triangles. The agreement is obvious.

CONCLUSIONS

For a heat conduction problem in a wedge enclosure of constant boundary temperature with or without phase change, it has been shown that the integrated heat flow rate over its arbitrarily chosen wedge surface r_1 at time t_1 equals the heat flow rate over its expanded surface r_i at time t_i, where

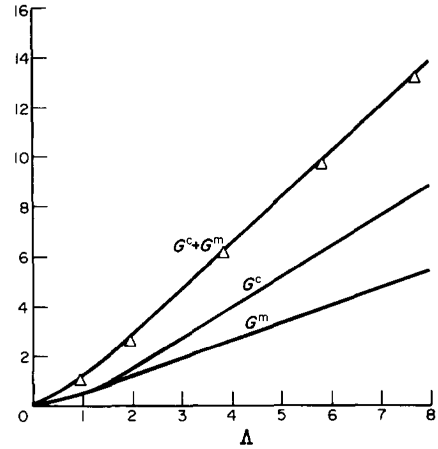

FIG. 1. Geometric factors G^c and G^m vs Λ ($\beta = 1, \theta_0 = \pi/2$ and $T_i^* = 0$).

$r_1/(4\alpha t_1)^{1/2} = r_2/(4\alpha t_2)^{1/2} = r_i/(4\alpha t_i)^{1/2} = \ldots = \Lambda$, and the proportionality Λ is the dimensionless wedge surface parameter.

The above similarity has led to the fact that the wedge invariant as defined by Wei and Berry [4] also exists for heat conduction with phase change.

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